

Progress in study of $\mathcal{N} = 4$ SYM effective action

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Abstract

We review the basic results concerning the structure of effective action in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in Coulomb phase. Various classical formulations of this theory are considered. We show that the low-energy effective action depending on all fields of $\mathcal{N} = 4$ vector multiplet can be exactly found. This result is discussed on the base of algebraic analysis exploring the general harmonic superspace techniques and on the base of straightforward quantum field theory calculations using the $\mathcal{N} = 2$ supersymmetric background field method. We study the one-loop effective action beyond leading low-energy approximation and construct supersymmetric generalization of Heisenberg-Euler-Schwinger effective action depending on all fields of $\mathcal{N} = 4$ vector multiplet. We also consider the derivation of leading low-energy effective action at two loops.

1 Introduction

Four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory possesses the remarkable properties in quantum domain (see e.g. the reviews [1], [2], [3]):

(i) $\mathcal{N} = 4$ SYM theory is the maximally extended rigid supersymmetric field model. It means, if $\mathcal{N} > 4$, the supermultiplet includes the fields with spins greater than 1. Consistent description of their interaction demands to take into account at least gravity, i.e. to localize the supersymmetry.

(ii) $\mathcal{N} = 4$ SYM theory is the finite four dimensional quantum field model. The theory under consideration has the only coupling (gauge coupling g) demanding no any infinite renormalization. The corresponding beta-function vanishes.

(iii) $\mathcal{N} = 4$ SYM theory is the superconformal invariant field model. The theory under consideration has no any scale. There are no explicit mass parameters in Lagrangian and no any scale is generated by radiative corrections.

These properties allows to treat $\mathcal{N} = 4$ SYM theory as the unique quantum field theory model.

Recently there were found unexpected links between $\mathcal{N} = 4$ SYM theory and string/brane theory. The $\mathcal{N} = 4$ SYM theory is closely related to $D3$ -branes (see e.g. the review [4]) and there exists a conjecture that $D3$ -brane interactions in static limit can be completely described in terms of effective action of $\mathcal{N} = 4$ SYM theory [5] [6]

invariant operators in $\mathcal{N} = 4$ SYM theory can be found on the base of low-energy effective action of type IIB superstring compactified on $AdS_5 \times S_5$ manifold and vice versa (see e.g. the reviews [7], [3]). Such properties allow to consider $\mathcal{N} = 4$ SYM theory as a part of superstring theory.

This paper is devoted to a brief review of problem of low-energy effective action in $\mathcal{N} = 4$ SYM theory. We begin with discussing the diverse formulations of the model. In particular, we consider the $\mathcal{N} = 1$ superfield formulation and $\mathcal{N} = 2$ harmonic superspace formulation. Both these formulations are used further for study of the various aspects of effective action. We show that the leading low-energy effective action depending on all fields of $\mathcal{N} = 4$ vector multiplet can be exactly found. Then we study the one-loop effective action, also depending on all fields of $\mathcal{N} = 4$ vector multiplet, beyond leading low-energy order. After that, we discuss the structure of leading low-energy effective action at two loops.

2 The various formulations of $\mathcal{N} = 4$ SYM theory

The $\mathcal{N} = 4$ SYM theory is a dynamical field model of $\mathcal{N} = 4$ vector multiplet. As known, the on-shell spectrum of $\mathcal{N} = 4$ vector multiplet consists of one vector A_m , six real scalars ϕ^I and four Majorana spinors λ^A (see e.g. [1]). At present, three formulations of $\mathcal{N} = 4$ SYM theory are usually used: component formulation, formulation in terms of $\mathcal{N} = 1$ superfields and formulation in terms of $\mathcal{N} = 2$ harmonic superfields.

Component formulation. Action of $\mathcal{N} = 4$ SYM theory in terms of component fields A_m, ϕ^I, λ^A has been constructed in [8] by means of dimensional reduction from ten dimensional $\mathcal{N} = 1$ SYM theory to four dimensions (see also [9] and the reviews [1], [2], [3]). This action is written as follows

$$S = \int d^4x \text{tr} \left\{ -\frac{1}{2g^2} F^{mn} F_{mn} + \frac{1}{2} \mathcal{D}_m \phi^I \mathcal{D}^m \phi^I - i \bar{\lambda}^A \sigma^m \mathcal{D}_m \lambda_A + \right. \\ \left. + g C_I^{AB} \lambda_A [\phi^I, \lambda_B] + g \bar{C}_{IAB} \bar{\lambda}^A [\phi^I, \bar{\lambda}^B] + \frac{1}{2} g^2 [\phi^I, \phi^J] [\phi^I, \phi^J] \right\}. \quad (1)$$

Here g is gauge coupling, C_I^{AB}, \bar{C}_{IAB} are six-dimensional analogs of four-dimensional sigma-matrices, the scalar and spinor fields belong to adjoint representation of the gauge group. One can show that action (1) is invariant under four hidden on-shell supersymmetries [1], [2], [3].

To describe a structure of ground state of the theory one considers the conditions of vanishing the scalar potential in action (1). It has the form $\text{tr}[\phi^I, \phi^J][\phi^I, \phi^J] = 0$. Here $\phi^I = \phi_c^I T^c$ where T^c are the generators of the gauge algebra in adjoint representation. Hence, the scalar potential vanishes for the fields ϕ_c^I satisfying the condition

$$\phi_{c_1}^I \phi_{c_2}^J \phi_{d_1}^I \phi_{d_2}^J \text{tr}([T^{c_1}, T^{c_2}][T^{d_1}, T^{d_2}]) = 0. \quad (2)$$

Let r is a rank of gauge group. Then for each fixed index I , the r components ϕ_c^I can have nonzero values. If all of them are nonzero, the gauge group is broken down to its maximal Abelian subgroup. In particular, if the gauge group is $SU(N)$, we get group $U(1)^{N-1}$ after symmetry breaking. This case is called Coulomb phase of the theory under consideration. For the rest of the $\mathcal{N} = 4$ SYM theory, the condition (2) is satisfied for

Formulation in terms of $\mathcal{N} = 1$ superfields. On-shell $\mathcal{N} = 4$ vector multiplet can be decomposed into a sum of on-shell $\mathcal{N} = 1$ vector multiplet and three scalar multiplets. Each of these $\mathcal{N} = 1$ multiplets is formulated in superfield terms on $\mathcal{N} = 1$ superspace. It means, there exists a possibility to formulate $\mathcal{N} = 4$ SYM theory in terms of real scalar $\mathcal{N} = 1$ superfield V and three chiral Φ^i and antichiral $\bar{\Phi}_i$ superfields. The corresponding action is written as follows [10]

$$S = \frac{1}{g^2} \text{tr} \left\{ \int d^4x d^2\theta W^2 + \int d^4x d^4\theta \bar{\Phi}_i e^V \Phi^i e^{-V} + \right. \\ \left. + \frac{1}{3!} \int d^4x d^2\theta i \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + \frac{1}{3!} \int d^4x d^2\bar{\theta} i \epsilon^{ijk} \bar{\Phi}_i [\bar{\Phi}_j, \bar{\Phi}_k] \right\}. \quad (3)$$

Here W_α is $\mathcal{N} = 1$ superfield strength, all superfields are taken in the adjoint representation of the gauge group. The action (3) possesses the manifest $\mathcal{N} = 1$ supersymmetry. However this action has three on-shell hidden global supersymmetries ([10], see also [26]) and as a result ones get $\mathcal{N} = 4$ supersymmetric theory.

Formulation in terms of $\mathcal{N} = 2$ harmonic superspace. Another way of considering the $\mathcal{N} = 4$ vector multiplet is to decompose it into a sum of $\mathcal{N} = 2$ vector multiplet and hypermultiplet. Both these $\mathcal{N} = 2$ multiplets can be formulated in terms of superfields defined on harmonic superspace [11], [12]. In harmonic superspace approach, the $\mathcal{N} = 2$ vector multiplet is described by analytic superfield V^{++} and the hypermultiplet is described by analytic superfield q^+ . It means, there can exist a formulation of the $\mathcal{N} = 4$ SYM theory in terms of the superfields V^{++}, q^+ . Such a formulation actually exists. The corresponding action is written as follows [12]

$$S = \frac{1}{2g^2} \text{tr} \int d^8z \mathcal{W}^2 - \frac{1}{2} \text{tr} \int d\zeta^{(-4)} q^{+a} (D^{++} + iV^{++}) q_a^+. \quad (4)$$

Here d^8z is integration measure over chiral subspace of general $\mathcal{N} = 2$ superspace, $d\zeta^{(-4)}$ is integration measure over analytic subspace of harmonic superspace, $q_a = (q^+, \tilde{q}^+)$, $q^a = \epsilon^{ab} q_b$ is the hypermultiplet analytic superfield in the adjoint representation of the gauge group and \mathcal{W} is the $\mathcal{N} = 2$ superfield strength of analytic superfield V^{++} . The action (4) is manifestly $\mathcal{N} = 2$ supersymmetric. However it possesses the hidden on-shell $\mathcal{N} = 2$ supersymmetry and as a result ones get $\mathcal{N} = 4$ supersymmetric theory.

3 Exact low-energy effective action.

Possibility of constructing the exact low-energy effective action in $\mathcal{N} = 4$ SYM theory is direct consequence of the strong restrictions imposed by maximally extended rigid supersymmetry, R -symmetry and scale invariance on a structure of effective action. For exploration of these restrictions it is convenient to use the formulation of the $\mathcal{N} = 4$ SYM theory in terms of harmonic superspace, where the most number of supersymmetries is manifest in compare with the component and $\mathcal{N} = 1$ superspace formulations. Of course, the same results, in principle, can be obtained using any of the formulations. The feature of harmonic superspace formulation is manifest $\mathcal{N} = 2$ supersymmetry which simplify a general consideration.

In harmonic superspace approach, the $\mathcal{N} = 4$ SYM effective action Γ depends on

invariance, the superfield V^{++} enters into effective action through the $\mathcal{N} = 2$ superfield strengths \mathcal{W} and $\bar{\mathcal{W}}$.¹ The effective action is $\Gamma[\mathcal{W}, \bar{\mathcal{W}}, q^+]$ has the following general form

$$\Gamma[\mathcal{W}, \bar{\mathcal{W}}, q^+] = S[V^{++}, q^+] + \bar{\Gamma}[\mathcal{W}, \bar{\mathcal{W}}, q^+]. \quad (5)$$

where $S[V^{++}, q^+]$ is the classical action (4) and $\bar{\Gamma}[\mathcal{W}, \bar{\mathcal{W}}, q^+]$ incorporates all quantum corrections. In low-energy approximation we neglect all spinor and space-time derivatives what allows to write the $\bar{\Gamma}[\mathcal{W}, \bar{\mathcal{W}}, q^+]$ in terms of effective Lagrangian $\mathcal{L}_{eff}(\mathcal{W}, \bar{\mathcal{W}}, q^+)$ as follows

$$\bar{\Gamma}[\mathcal{W}, \bar{\mathcal{W}}, q^+] = \int d^{12}z du \mathcal{L}_{eff}(\mathcal{W}, \bar{\mathcal{W}}, q^+). \quad (6)$$

Here $d^{12}z$ is the full $\mathcal{N} = 2$ superspace measure and du means integration measure over harmonics. We denote

$$\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}) = \mathcal{L}_{eff}(\mathcal{W}, \bar{\mathcal{W}}, q^+)|_{q^+=0}. \quad (7)$$

The quantity $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ determines the $\mathcal{N} = 4$ SYM low-energy effective action in sector of $\mathcal{N} = 2$ vector multiplet and is called the nonholomorphic effective potential. As a result the effective Lagrangian can be presented in the form

$$\mathcal{L}_{eff}(\mathcal{W}, \bar{\mathcal{W}}, q^+) = \mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}) + \mathcal{L}_q(\mathcal{W}, \bar{\mathcal{W}}, q^+). \quad (8)$$

where the quantity $\mathcal{L}_q(\mathcal{W}, \bar{\mathcal{W}}, q^+)$ vanishes at $q^+ = 0$ and determines the $\mathcal{N} = 4$ SYM low-energy effective action in the hypermultiplet sector. Further we show that the effective Lagrangian \mathcal{L}_{eff} can be exactly found in Coulomb phase where \mathcal{W} and q^+ belong to maximal Abelian subgroup of the gauge group.

We begin with $SU(2)$ gauge theory spontaneously broken down to $U(1)$. First of all, ones consider the calculation of nonholomorphic effective potential following the work [13]. Taking into account the mass dimensions of $d^{12}z$, \mathcal{W} and $\bar{\mathcal{W}}$ ones see that nonholomorphic effective potential is dimensionless, and hence

$$\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}) = \mathcal{H}\left(\frac{\mathcal{W}}{\Lambda}, \frac{\bar{\mathcal{W}}}{\Lambda}\right). \quad (9)$$

with Λ be some scale. Now we use the conditions of R -invariance and scale independence of $\mathcal{N} = 4$ SYM quantum theory. It leads to equation

$$\Lambda \frac{d}{d\Lambda} \int d^{12}z \mathcal{H}\left(\frac{\mathcal{W}}{\Lambda}, \frac{\bar{\mathcal{W}}}{\Lambda}\right) = 0. \quad (10)$$

As shown in [13], the equation (10) has the only solution

$$\mathcal{H}\left(\frac{\mathcal{W}}{\Lambda}, \frac{\bar{\mathcal{W}}}{\Lambda}\right) = c \ln \frac{\mathcal{W}}{\Lambda} \ln \frac{\bar{\mathcal{W}}}{\Lambda}, \quad (11)$$

where c is an arbitrary dimensionless constant which can depend only on gauge coupling g . Result (11) is exact. Thus, we see that all quantum corrections, perturbative or non-perturbative, are included into a single constant c . Moreover, it was argued in the work

¹We imply that the quantum theory is formulated within background field method leading to gauge invariant effective action (a formulation of background field method with $\mathcal{N} = 2$ supersymmetry is given in [14]).

[13] (see also [14]) that the constant c really gets only one-loop contribution.² The direct calculations of the constant c were carried out in the papers [15], [16], [17], [14] (see also the review [18] and calculations of the constant c in $\mathcal{N} = 2$ superconformal theories in [19], [33]). Thus, the exact low-energy effective action in the $\mathcal{N} = 2$ vector multiplet sector is known. However, to get complete effective Lagrangian $\mathcal{L}_{eff}(\mathcal{W}, \bar{\mathcal{W}}, q^+)$ we have to find the hypermultiplet dependent part $\mathcal{L}_q(\mathcal{W}, \bar{\mathcal{W}}, q^+)$ in (8).

Problem of complete \mathcal{L}_{eff} has been solved in works [22], [23]. We consider the calculation of \mathcal{L}_q following the paper [22]. First of all, we note that the effective action (7) is manifestly $\mathcal{N} = 2$ supersymmetric. However, the classical action (4) is also invariant under the hidden $\mathcal{N} = 2$ supersymmetry, forming together with manifest supersymmetry, the complete $\mathcal{N} = 4$ supersymmetry of $\mathcal{N} = 4$ SYM theory. We will seek for the function \mathcal{L}_q demanding that the full effective action (7) is invariant under the hidden $\mathcal{N} = 2$ supersymmetry transformations. One can show, it is sufficient to consider the Abelian background superfields $\mathcal{W}, \bar{\mathcal{W}}, q^+$ satisfying the free classical equations of motion [22]. In this case, the hidden $\mathcal{N} = 2$ supersymmetry transformations have the form [12]

$$\delta\mathcal{W} = \frac{1}{2}\epsilon^{\dot{\alpha}a}\bar{D}_{\dot{\alpha}}^-q_a^+, \delta\bar{\mathcal{W}} = \frac{1}{2}\epsilon^{\alpha a}D_{\alpha}^-q_a^+, \delta q_a^+ = \frac{1}{4}(\epsilon_a^{\beta}D_{\beta}^+\mathcal{W} + \bar{\epsilon}_a^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}^+\bar{\mathcal{W}}). \quad (12)$$

We demand

$$\delta \int d^{12}z du (\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}) + \mathcal{L}_q(\mathcal{W}, \bar{\mathcal{W}}, q^+)) = 0. \quad (13)$$

where the transformation δ in (13) is generated by (12) and nonholomorphic effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ is given by (9), (11). The equation (13) has been analysed in [22]. The function \mathcal{L}_q was written as a power series in quantity $\frac{q^{+a}q_a^-}{\mathcal{W}\bar{\mathcal{W}}}$ with unknown coefficients which were found from equation (13). It was proved that there exists the only function $\mathcal{L}_q(\mathcal{W}, \bar{\mathcal{W}}, q^+)$ forming, together with given function $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ (9), (11), a solution to the equation (13). Final result has the form

$$\mathcal{L}_{eff} = c(\ln\frac{\mathcal{W}}{\Lambda}\ln\frac{\bar{\mathcal{W}}}{\Lambda} + (X - 1)\frac{\ln(X - 1)}{X} + (\text{Li}_2(X) - 1)). \quad (14)$$

where $\text{Li}_2(X)$ is Euler dilogarithm function, c is the same constant as in (11) and

$$X = -\frac{q^{ia}q_{ia}}{\mathcal{W}\bar{\mathcal{W}}}. \quad (15)$$

Here on-shell $\mathcal{N} = 2$ superfield q^{ia} is defined by $q^{+a} = q^{ia}u_i^+$ [12]. It means that the quantity X (15) is harmonic independent. Therefore \mathcal{L}_{eff} does not depend on harmonics and the integral over harmonics in (6) can be omitted.

We emphasize that effective Lagrangian (14) determines the exact $\mathcal{N} = 4$ supersymmetric low-energy effective action depending on all fields of $\mathcal{N} = 4$ vector multiplet. Nonholomorphic effective potential (9), (11) is exact [13]. The function \mathcal{L}_q was uniquely found on the base of this potential using invariance of effective action under the hidden $\mathcal{N} = 2$ supersymmetry transformations. Therefore the effective Lagrangian (14) determines the exact low-energy effective action of $\mathcal{N} = 4$ SYM theory with the $SU(2)$ gauge group spontaneously broken down to $U(1)$.

²The constant c is calculated in [15] and [16] and found to be $c = \frac{1}{2} \ln \frac{1}{\Lambda^2}$ [15].

We investigate now a component structure of low-energy effective action in bosonic sector. Let the on-shell superfield \mathcal{W} has the following bosonic components: complex scalar ϕ and self-dual spinor component of Abelian strength $F^{\alpha\beta}$ and the on-shell hypermultiplet superfield q^{ia} has the scalar component f^{ia} . Substituting the effective Lagrangian (14) depending on above superfields into effective action (6) and integrating over anticommuting coordinates one gets (see the details in [22])

$$\bar{\Gamma} = 4c \int d^4x \frac{F^2 \bar{F}^2}{(|\phi|^2 + f^{ia} f_{ia})^2}. \quad (16)$$

Expression in the denominator is the invariant quadratic combination of six scalars from $\mathcal{N} = 4$ vector multiplet under R -symmetry group of $\mathcal{N} = 4$ supersymmetry. We could expect such a result from the very beginning. It means, in particular, that starting with bosonic component effective action in sector of $\mathcal{N} = 2$ vector multiplet and demanding the invariance under R -symmetry group we could restore uniquely the complete result (16).

The exact low-energy effective Lagrangian (14) has been obtained for $SU(2)$ gauge group spontaneously broken down to $U(1)$. Generalization for the $SU(N)$ group spontaneously broken down to $U(1)^{N-1}$ can be done simply enough using the procedure described in [17]. The final result is presented as a sum over roots of gauge algebra of the terms having the form (14) (see the details in [22]). No any other contributions to low-energy effective action in $\mathcal{N} = 4$ SYM theory are possible.³ It leads to important conclusion that we get the exact low-energy effective action in theory with $SU(N)$ gauge group spontaneously broken down to $U(1)^{N-1}$ group for any N .

4 Supergraph calculation of the effective Lagrangian

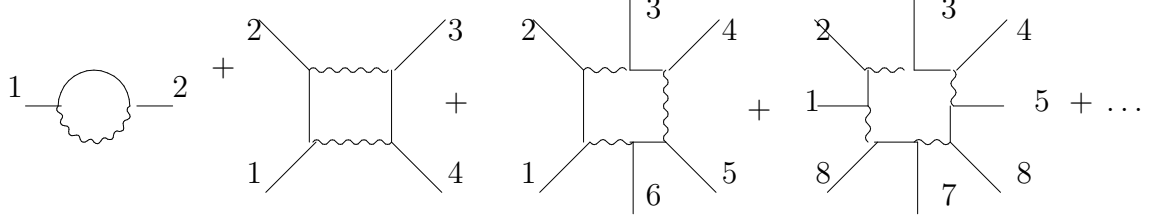
We have constructed the exact $\mathcal{N} = 4$ supersymmetric low-energy effective action from purely algebraic consideration. One can show that this result can be obtained in framework of quantum field theory by calculating the one-loop harmonic supergraphs with an arbitrary number of the external hypermultiplet legs on the background of constant superfields \mathcal{W} , $\bar{\mathcal{W}}$ [23]. We discuss now the basic elements of this calculation.

The $\mathcal{N} = 4$ SYM theory is formulated in $\mathcal{N} = 2$ harmonic superspace in terms of $\mathcal{N} = 2$ gauge superfield and hypermultiplet superfield. We consider the theory with gauge group $SU(2)$ spontaneously broken down to $U(1)$. To provide the gauge invariance of the quantum theory we use the manifestly $\mathcal{N} = 2$ supersymmetric background field methods [20]. In this case the superfields V^{++} , q^+ are splitted into background and quantum parts. All quantum superfield propagators depend on Abelian background superfields \mathcal{W} , $\bar{\mathcal{W}}$ and, as a result, the effective action is gauge invariant and manifestly $\mathcal{N} = 2$ supersymmetric functional of the \mathcal{W} , $\bar{\mathcal{W}}$ and background hypermultiplet superfields.

Effective Lagrangian \mathcal{L}_{eff} (8) is determined by nonholomorphic effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ and hypermultiplet dependent effective Lagrangian \mathcal{L}_q . Nonholomorphic effective potential has been studied in enough details (see the review [18]). We consider here the harmonic supergraph calculation of the \mathcal{L}_q .

³In principle, if we take into account only manifest $\mathcal{N} = 2$ supersymmetry and structure of $SU(N)$ group with $N > 2$, then there can be the terms in low-energy effective action different from (14) [24].

The hypermultiplet dependent contributions to the one-loop effective action are presented by following infinite sequence of the supergraphs:



Here the wavy line stands for $\mathcal{N} = 2$ gauge superfield propagator and solid external and internal lines stand for external background hypermultiplet superfields and quantum hypermultiplet propagators respectively. Explicit expressions for the background dependent propagators are given in [20], [23]. In the case under consideration it is sufficient to consider the constant background strengths $\mathcal{W}, \bar{\mathcal{W}}$. Then any of the above supergraphs can be exactly calculated. The supergraph with $2n, (n = 1, 2, \dots)$ external hypermultiplet legs has the form

$$\Gamma_{(2n)} = \frac{1}{n^2(n+1)} \frac{1}{(4\pi)^2} \int d^{12}z X^n, \quad (17)$$

where X is given by (15). To find the complete dependence of effective action on hypermultiplets, we have to sum all $\Gamma_{(2n)}$ (17). It leads to $\mathcal{L}_q = c((X-1)\frac{\ln(X-1)}{X} + (\text{Li}_2(X)-1))$ with $c = \frac{1}{(4\pi)^2}$. This result exactly corresponds to (14).

5 One-loop effective action beyond leading low-energy approximation

Eqs (14), (15) determine the effective action in leading low-energy order. In this section we consider a structure of the one-loop effective action beyond leading approximation [26] using formulation (3) of $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 1$ superfields. Although such a formulation preserves lesser number of manifest supersymmetries than harmonic superspace approach, it has some positive features of technical character. In particular, the one-loop effective action in $\mathcal{N} = 1$ superfield theories can be studied by well developed operator methods (see e.g. [28]).⁴

As well as in previous section, we consider the theory with gauge group $SU(2)$ spontaneously broken down to $U(1)$. Generalization for $SU(N)$ group spontaneously broken down to $U(1)^{N-1}$ can be done using procedure described in [17]. To provide gauge invariance of effective action we use $\mathcal{N} = 1$ background field method [28], [10] and $\mathcal{N} = 1$ supersymmetric R_ξ -gauges [31].

Background field method begins with splitting the fields into background and quantum. First of all ones rename the fields in action (3) as follows: $\Phi^1 = \Phi$, $\Phi^2 = Q$, $\Phi^3 = \tilde{Q}$. The $\mathcal{N} = 1$ chiral superfield Φ contains two real scalars corresponding to $\mathcal{N} = 2$ vector multiplet and the $\mathcal{N} = 1$ chiral superfields Q, \tilde{Q} contain four real scalars corresponding to hypermultiplet. After background-quantum splitting ones get a theory of quantum superfields v, ϕ, q, \tilde{q} in the background fields V, Φ, Q, \tilde{Q} . In one-loop approximation the effective

⁴For the definition of the background field method and the effective action in $\mathcal{N} = 2$ superspace see [20].

action is determined on the base of quadratic part (in quantum fields) of classical action with gauge fixing term and ghost action. It allows to obtain the one-loop quantum correction to classical action in the form (see the details in [26])

$$\bar{\Gamma} = i\text{Tr} \ln(\square - iW^\alpha \nabla_\alpha - i\bar{W}^{\dot{\alpha}} \bar{\nabla}_{\dot{\alpha}} - M). \quad (18)$$

$$M = \bar{\Phi}\Phi + \bar{Q}Q + \bar{\tilde{Q}}\tilde{Q}. \quad (19)$$

Here $W^\alpha, \bar{W}^{\dot{\alpha}}$ are the background strengths. Tr means the functional trace of the operator acting in space of quantum superfields v .

Effective action (18) has been calculated for the $\mathcal{N} = 1$ supersymmetric background corresponding to constant Abelian strength F_{mn} and constant scalar field ϕ in [29] (see also [25]) using superfield proper-time techniques. This calculation corresponds to the case when the superfields Q, \tilde{Q} vanish. However, since these superfields enter in effective action in special R -symmetry invariant combination M (19), the effective action for background with constant component fields F_{mn}, ϕ, f^{ia} is obtained from result given in [29], [25] replacing the expression $\bar{\Phi}\Phi$ in [29], [25] by M (19). The final result for the effective action (18) is written as follows

$$\bar{\Gamma} = \frac{1}{8\pi^2} \int d^8z \int_0^\infty dt e^{-t} \frac{W^2 \bar{W}^2}{M^2} \omega(t\Psi, t\bar{\Psi}). \quad (20)$$

where

$$\bar{\Psi}^2 = \frac{1}{M^2} D^2 W^2, \quad \Psi^2 = \frac{1}{M^2} \bar{D}^2 \bar{W}^2. \quad (21)$$

The function $\omega(x, y)$ has been introduced in [29] and looks like

$$\omega(x, y) = \frac{\cosh(x) - 1}{x^2} \frac{\cosh(y) - 1}{y^2} \frac{x^2 - y^2}{\cosh(x) - \cosh(y)}. \quad (22)$$

Eq (20) determines the one-loop $\mathcal{N} = 4$ SYM effective action in $\mathcal{N} = 1$ superfield form for constant $\mathcal{N} = 1$ supersymmetric background.

The effective action (20) includes dependence on arbitrary powers of Abelian strength F_{mn} , not only on fourth power like in (14), and, hence, determines the one-loop effective action beyond leading low-energy approximation. It is given by integral over proper-time t , contains background superfields corresponding to all fields of $\mathcal{N} = 4$ vector multiplet, both bosonic and fermionic. Therefore this effective action can be considered as a generalization of Heisenberg-Euler-Schwinger effective action for $\mathcal{N} = 4$ SYM theory. In component form for the constant bosonic background F_{mn} , the one-loop effective action of $\mathcal{N} = 4$ SYM theory was obtained in [32]. Expanding the function $\omega(x, y)$ (22) in power series in x, y we get the expansion of effective action (20) over $\frac{F^2}{M^2}$. It is interesting to note, this expansion does not include the F^6 term, that is a feature just of $\mathcal{N} = 4$ SYM theory [32], [29]. However, such a term appears in expansion of effective action of arbitrary $\mathcal{N} = 2$ superconformal model [29].

The effective action (20) is given in manifestly $\mathcal{N} = 1$ supersymmetric form. It was shown in [26] that each term in expansion of this effective action (20) over $\frac{F^2}{M^2}$ can be written in manifestly $\mathcal{N} = 2$ supersymmetric form. To do that ones introduce the $\mathcal{N} = 2$ superconformal scalars [29]

Using the expansion of function $\omega(x, y)$ (22), the $\mathcal{N} = 1$ superfields (21) and applying the procedure of restoring the $\mathcal{N} = 2$ strength superfields $\mathcal{W}, \bar{\mathcal{W}}$ ones get manifestly $\mathcal{N} = 2$ supersymmetric expansion of effective action (20) in power series in the quantities (23)

$$\bar{\Gamma} = \Gamma^{(0)} + \Gamma^{(2)} + \Gamma^{(3)} + \dots \quad (24)$$

Here the $\Gamma^{(n)}$ contains the terms $\Psi^m \bar{\Psi}^{n-m}$ what corresponds to $\frac{F^{4+2n}}{M^{2+2n}}$ in bosonic component sector. A few first terms in the expansion (24) can be easily found. Leading low-energy contribution is

$$\Gamma^{(0)} = \frac{1}{(4\pi)^2} \int d^{12}z \left(\ln \frac{\mathcal{W}}{\Lambda} \ln \frac{\bar{\mathcal{W}}}{\Lambda} + \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} X^k \right), \quad (25)$$

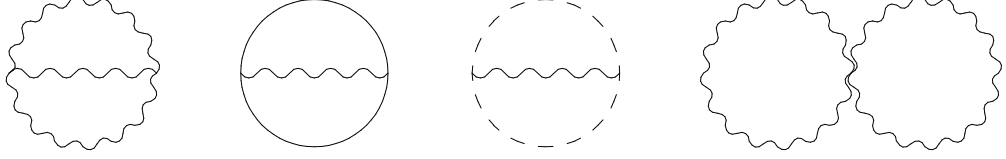
where X is given by (15). The series in right hand side of (25) can be summed up and the result is the hypermultiplet dependent effective Lagrangian $\mathcal{L}_q = \frac{1}{(4\pi)^2} ((X-1) \frac{\ln(X-1)}{X} + (\text{Li}_2(X)-1))$ in (8). Thus, the $\Gamma^{(0)}$ (25) coincides with exact low-energy effective action (6), (14). The terms $\Gamma^{(2)}, \Gamma^{(3)}, \dots$ present the subleading corrections to leading contribution (25). The explicit expressions for $\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}$ were calculated in [26] (see also [27]).

6 Low-energy effective action at two loops

In this section we consider a structure of leading two-loop contribution to effective action. As we discussed in Section 3, the exact low-energy effective action contains F^4 term in bosonic component sector and should originate only from one loop.⁵ It means, that two-loop correction to classical action should begin at least with F^6 term. The corresponding calculations have been done in [6]. The main motivation of the work [6] was to study the relation between $\mathcal{N} = 4$ SYM effective action and $D3$ -brane interactions in superstring theory. In static limit this interaction is described by Born-Infeld action which can be presented as a series in powers of Abelian strength F_{mn} (see the review [30]). It was proved in [6] that in 't Hooft limit the coefficient at two-loop F^6 term in $\mathcal{N} = 4$ SYM effective action exactly coincides with the coefficient at F^6 term in expansion of Born-Infeld action. Coincidence of the coefficients at F^4 terms was earlier demonstrated in [29]. These results confirm the conjecture that $D3$ -brane interactions can be completely described in terms of $\mathcal{N} = 4$ SYM theory. We discuss further the basic elements of calculating the leading two-loop contribution to effective action in 't Hooft limit using the harmonic superspace formulation of $\mathcal{N} = 4$ SYM theory.

In $\mathcal{N} = 2$ harmonic superspace the $\mathcal{N} = 4$ SYM theory is formulated in terms of $\mathcal{N} = 2$ gauge superfield and hypermultiplet superfield. We consider $\mathcal{N} = 4$ SYM theory with gauge group $SU(N+1)$ spontaneously broken down to $SU(N) \times U(1)$ and study the effective action depending on the background superfields $\mathcal{W}, \bar{\mathcal{W}}$ belonging to Abelian factor $U(1)$. The effective action is calculated on the base of $\mathcal{N} = 2$ supersymmetric background field method [20] in large N limit. Two-loop contribution to effective action is given by the following harmonic supergraphs

⁵Recently it was shown [34] that unlike $\mathcal{N} = 4$ SYM theory, F^4 term can appear in two-loop effective action for $\mathcal{N} = 2$ superconformal field theories.



where the wavy, solid and dashed lines stand for the propagators of $\mathcal{N} = 2$ gauge, hypermultiplet and ghost superfields respectively. All propagators depend on Abelian on-shell background superfields $\mathcal{W}, \bar{\mathcal{W}}$. Explicit forms of the propagators are given in [20], [6]. In the above supergraphs we expand each propagator in $\mathcal{W}, \bar{\mathcal{W}}$ and their spinor derivatives and look for the leading low-energy contribution. One can show that the expected manifestly $\mathcal{N} = 2$ supersymmetric F^6 term is generated by the terms in effective action containing the fourth order in $D_\alpha^i \mathcal{W}$ and second order in $\bar{D}_{i\dot{\alpha}} \bar{\mathcal{W}}$ (plus the conjugate terms). It was proved in [6] that the supergraphs including the hypermultiplet and ghost lines and supergraph with quartic gauge superfield vertex do not contribute to effective action in leading low-energy approximation. Thus, we have to study the only supergraph with two cubic gauge superfield vertices. Calculation of this supergraph is done using the standard D -algebra manipulations. The final result for two-loop correction $\Gamma_2[\mathcal{W}, \bar{\mathcal{W}}]$ to effective action has the form

$$\Gamma_2[\mathcal{W}, \bar{\mathcal{W}}] = c_2 g^2 N^2 \int d^{12}z \frac{1}{\mathcal{W}^2} \ln \frac{\mathcal{W}}{\Lambda} D^4 \ln \frac{\mathcal{W}}{\Lambda} + h.c. \quad (26)$$

where the $c_2 = \frac{1}{24(4\pi)^4}$.⁶ Eq (26) determines the leading two-loop low-energy contribution to effective action. The coefficient c_2 exactly corresponds to the coefficient at F^6 term in expansion of Born-Infeld action [6].

7 Summary

We have considered the current state of problem of effective action in $\mathcal{N} = 4$ SYM theory. The main results are formulated as follows:

1. Low-energy effective action depending on all fields of $\mathcal{N} = 4$ vector multiplet is exactly found.
2. One-loop effective action depending on all fields of $\mathcal{N} = 4$ vector multiplet is exactly found for the supersymmetric background corresponding to constant Abelian strength F_{mn} .
3. Leading two-loop contribution to low-energy effective action containing F^6 term in bosonic component sector is found.
4. The coefficients at classical F^2 term, one-loop F^4 term and two-loop F^6 term in effective action exactly coincide with the corresponding coefficients in expansion of Born-Infeld action.

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⁶The coefficient c_2 is identical to the coefficient of F^6 term in expansion of Born-Infeld action.

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